113 Class Problems: Actions and Permutation Groups

1. Prove that for $n \in \mathbb{N}$, $|Sym_n| = n!$.

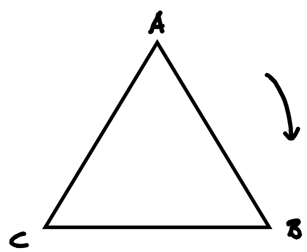
   Solution:

Let $f \in Sym_n$. f is determined by $\{f(1), f(2), \ldots, f(n)\}$

As f is a bijection there are n choices for $f(1)$, $n-1$ choices for $f(2), \ldots$. $\Rightarrow |Sym_n| = n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$

2. Let $S = \{A, B, C\}$ where A, B and C are the vertices of an equilateral triangle. Define an action of $\mathbb{Z}/6\mathbb{Z}$ on as follows:

   Given $[a] \in \mathbb{Z}/6\mathbb{Z}$ and $X \in S$, $[a](X)$ = clockwise rotation of X about the center of the triangle by angle $\frac{2\pi a}{3}$. For example:



$\frac{2\pi}{3} = [1] \Rightarrow$

$[1](A) = B$

$[1](B) = C$

$[1](C) = A$

   (a) Prove that this action is well defined, i.e if $a \equiv b \bmod 6$, then $[a](X) = [b](X)$ for all $X \in S$. Prove it is an action.

   (b) Is this action faithful?

   Solution:

a) $[a] = [b] \Rightarrow a = b + 6q, \; q \in \mathbb{Z}$

$[a](X) = \text{rotation of } X \text{ by } \frac{2\pi a}{3} = \frac{2\pi(b+6q)}{3} = \frac{2\pi b}{3} + 4\pi q$ integer multiple of 2π

$= \text{rotation of } X \text{ by } \frac{2\pi b}{3}$

$= [b](X)$

1/ $[0](X) = X \quad \forall X \in S$

2/ $([a]+[b])(X) = [a+b](X) = \text{Rotation of } X \text{ by } \frac{2\pi(a+b)}{3}$

$= \text{Rotation of } X \text{ by } \frac{2\pi b}{3}, \text{ follow by rotation by } \frac{2\pi a}{3}$

$= [a]([b](X))$

b) No $[0](X) = X = [3](X) \quad \forall X \in S$, however $[0] \neq [3]$

3. Prove that there is no faithful action of $\mathbb{Z}/11\mathbb{Z}$ on the set $\{1,2,3,4,5\}$.

Solution:

If $\phi : \mathbb{Z}/_{11\mathbb{Z}} \longrightarrow Sym_5$ was faithful $\Rightarrow \phi$ injective

$\Rightarrow \mathbb{Z}/_{11\mathbb{Z}} \cong Im(\phi) \Rightarrow |Im(\phi)| = 11$

$Im(\phi) \subset Sym_5$ is a subgroup, hence $|Im(\phi)| \,\big|\, |Sym_5|$

However $11 \nmid 5!$, hence no such action exists

4. Prove that if $|S| > 2$, $\Sigma(S)$ is non-Abelian.

Solution:

Let $a, b, c \in S$ such that $a \neq b$, $a \neq c$, $b \neq c$

Let $f, g \in \Sigma(S)$ be defined as follows:

$$f(x) = \begin{cases} a & \text{if } x = b \\ b & \text{if } x = a \\ x & \text{if } x \neq a, x \neq b \end{cases} \qquad g(x) = \begin{cases} c & \text{if } x = b \\ b & \text{if } x = c \\ x & \text{if } x \neq c, x \neq b \end{cases}$$

Consider $(f \circ g)(a)$ and $(g \circ f)(a)$

$(f \circ g)(a) = f(g(a)) = f(a) = b$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \neq \qquad \Rightarrow f \circ g \neq g \circ f$

$(g \circ f)(a) = g(f(a)) = f(b) = c$